On Ternary Quadratic Diophantine Equation $x^2 + y^2 = 17z^2$

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Abstract – The quadratic diophantine equation with three unknowns represented by $x^2 + y^2 = 17z^2$ is analyzed for finding its non-zero distinct integral solutions. Different patterns of solutions of the equation under consideration are obtained. A few interesting properties among the solutions are presented.

Index Terms – Ternary quadratic equation with three unknowns, integral solutions, polygonal numbers and pyramidal numbers.

1. INTRODUCTION

The quadratic diophantine equation with three unknowns offers an unlimited field for research because of their variety [1-3]. In particular, one may refer [4-19] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $x^2 + y^2 = 17z^2$ representing homogeneous quadratic diophantine equation with three unknowns for determining its infinitely many non-zero integral solutions. A few interesting properties among its solutions are given. Also, formulas for generating sequences of integer solutions based on its given solution are presented.

2. NOTATION

Polygonal number of rank n with size m

$$t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$$

Centered Hexagonal pyramidal number of rank n

$$Cp_{n,6} = n^3$$

Pronic number of rank n

$$\Pr_n = n(n+1)$$

Gnomonic number of rank n

$$GNO_n = 2n-1$$

Star number of rank n

$$S_n = 6n^2 - 6n + 1$$

3. METHOD OF ANALYSIS

The ternary quadratic diophantine equation to be solved for its non-zero distinct integral solution is

$$x^2 + y^2 = 17z^2 \quad (1)$$

Different patterns of solution of (1) are presented below.

3.1.PATTERN- I

Write 17 as

$$17 = (4+i)(4-i) \quad (2)$$

Assume

$$z = a^2 + b^2 \quad (3)$$

where a, b are non-zero distinct integers.

Using (2) and (3) in (1), we get

$$x^2 + y^2 = (4+i)(4-i)(a^2+b^2)^2$$

Employing the method of factorization, we have

$$(x+iy)(x-iy) = (4+i)(4-i)(a+ib)^2(a-ib)^2$$

Equating the positive and negative factors, we get

$$x + iy = (4+i)(a+ib)^2 \quad (4)$$

$$x - iy = (4-i)(a-ib)^2 \quad (5)$$

Equating the real and imaginary part either in (4) or (5), we get

$$x(a,b) = 4a^2 - 4b^2 - 2ab$$

$$y(a,b) = a^2 - b^2 + 8ab \quad (6)$$

Thus (6) and (3) represents non-zero distinct integral solutions of (1)

PROPERTIES :

- $y(a,a^2)+z(a,a^2)-2t_{4,a}-8Cp_{a,6}=0$
- $x(a,a+1)+y(a,a+1)+z(a,a+1)-8t_{4,a}+GNo_a+5=0$
- $9z(a,a)+x(a,a)$ is a perfect square.
- $x(a,a^2)-y(a,a^2)-z(a,a^2)-2(t_{4,a})^2+4t_{4,a}+10Cp_{a,6}=0$
- $y(a,a)+z(a,a)-t_{4,a}=0$

REMARK:

Write 17 as

$$17=(1+4i)(1-4i) \tag{7}$$

where a, b are non-zero distinct integers,

Using (7) and (3) in (1), we get

$$x^2+y^2=(1+4i)(1-4i)(a^2+b^2)^2$$

Employing the method of factorization, we have

$$(x+iy)(x-iy)=(1+4i)(1-4i)(a+ib)^2(a-ib)^2$$

Equating the positive and negative factors, we get

$$x+iy=(1+4i)(a+ib)^2 \tag{8}$$

$$x-iy=(1-4i)(a+ib)^2 \tag{9}$$

Equating the real and imaginary part either in (8) or (9), we get

$$\left.\begin{aligned} x(a,b)&=a^2-b^2-8ab \\ y(a,b)&=4a^2-4b^2+2ab \end{aligned}\right\} \tag{10}$$

Thus (10) and (3) represents non-zero distinct integral solutions of (1)

3.2.PATTERN II

Observe that (1) is written as

$$x^2+y^2=16z^2+z^2$$

$$\frac{x-4z}{z+y}=\frac{z-y}{x+4z}=\frac{\alpha}{\beta}\ ,\ \beta\neq 0 \tag{11}$$

which is equivalent to the system of double equations

$$\left.\begin{aligned} \beta x+\alpha y-(4\beta+\alpha)z&=0 \\ -\alpha x-\beta y+(\beta-4\alpha)z&=0 \end{aligned}\right\} \tag{12}$$

Solving (12) by applying the method of cross multiplication, the corresponding non-zero distinct integral solutions to (1) are obtained as

$$x(\alpha,\beta)=4\alpha^2-4\beta^2-2\alpha\beta$$

$$y(\alpha,\beta)=\alpha^2-\beta^2+8\alpha\beta$$

$$z(\alpha,\beta)=-\alpha^2-\beta^2$$

PROPERTIES **:**

- $x(1,\beta)+z(1,\beta)+3t_{4,\beta}+2pr_\beta-3=0$
- $x(\alpha,1)-t_{10,\alpha}-pr_\alpha+t_{4,\alpha}+4=0$
- $4y(\alpha,\alpha+1)-x(\alpha,\alpha+1)-34pr_\alpha=0$
- $4y(\alpha,\alpha-1)-x(\alpha,\alpha-1)-t_{70,\alpha}-pr_\alpha+t_{4,\alpha}=0$
- $z(\alpha,\alpha)+2t_{4,\alpha}=0$

REMARK:

In addition to (11), (1) may also be expressed in the form of ratio as

$$\frac{x-4z}{z-y}=\frac{z+y}{x+4z}=\frac{\alpha}{\beta},\beta\neq 0$$

Following the procedure as presented above, the corresponding non-zero distinct integral solutions to (1) is given by

$$x(\alpha,\beta)=-4\alpha^2+4\beta^2+2\alpha\beta$$

$$y(\alpha,\beta)=\alpha^2-\beta^2+8\alpha\beta$$

$$z(\alpha,\beta)=\alpha^2+\beta^2$$

3.3.PATTERN III

Introducing the linear transformations

$$x=u+v,\ y=u-v,\ z=2w \tag{13}$$

in (1), it is written as

$$u^2+v^2=34w^2 \tag{14}$$

Assume

$$w = c^2 + d^2 \qquad (15)$$

$$34 = (3+5i)(3-5i) \qquad (16)$$

Substituting (15) and (16) in (14), we get

$$(u+iv)(u-iv) = (3+5i)(3-5i)(c+id)^2(c-id)^2$$

Equating the positive and negative parts, we get

$$(u+iv) = (3+5i)(c+id)^2 \qquad (17)$$

$$(u-iv) = (3-5i)(c-id)^2 \qquad (18)$$

Equating the real and imaginary parts either in (17) or (18), we get

$$\left.\begin{aligned} u(c,d) &= 3c^2 - 3d^2 - 10cd \\ v(c,d) &= 5c^2 - 5d^2 + 6cd \end{aligned}\right\} \qquad (19)$$

Substituting (19) and (15) in (14), the corresponding non-zero integral solution to (1) are given by

$$x(c,d) = 8c^2 - 8d^2 - 6cd$$

$$y(c,d) = -2c^2 + 2d^2 - 16cd$$

$$z(c,d) = 2c^2 + 2d^2$$

PROPERTIES:

- $y(d,d+1) + z(d,d+1) + t_{18,d} + pr_d - t_{4,d} + 4 = 0$
- $x(c,1) + y(c,1) + z(c,1) + 13t_{4,c} - t_{18,c} - 13pr_c + 4 = 0$
- $x(c-1,c) + y(c-1,c) - 20t_{4,c} + 8pr_c - 6 = 0$
- $6z(1,1)$ is a nasty number.
- $y(c^2,c) + z(c^2,c) - 4t_{4,c} + 16Cp_{c,6} = 0$

REMARK:

Write 34 as

$$34 = (5+3i)(5-3i) \qquad (20)$$

Substituting (15) and (20) in (14), we get

$$(u+iv)(u-iv) = (5+3i)(5-3i)(c+id)^2(c-id)^2$$

Equating the positive and negative parts, we get

$$(u+iv) = (5+3i)(c+id)^2 \qquad (21)$$

$$(u-iv) = (5-3i)(c-id)^2 \qquad (22)$$

Equating the real and imaginary parts either in (21)or (22), we get

$$\left.\begin{aligned} u(c,d) &= 5c^2 - 5d^2 - 6cd \\ v(c,d) &= 3c^2 - 3d^2 + 10cd \end{aligned}\right\} \qquad (23)$$

Substituting (20) and (23) in (14), the corresponding non-zero integral solution to (1) are given by

$$x(c,d) = 8c^2 - 8d^2 - 4cd$$

$$y(c,d) = 2c^2 - 2d^2 - 16cd$$

$$z(c,d) = 2c^2 + 2d^2$$

3.4.PATTERN IV

Introducing the linear transformations,

$$x = u+v,\ y = u-v,\ z = 2w \qquad (24)$$

in (1), it is written as

$$u^2 - 25w^2 = 9w^2 - v^2$$

(1) can be written in the form of ratio as

$$\frac{u-5w}{3w-v} = \frac{3w+v}{u+5w} = \frac{\alpha}{\beta},\ \beta \neq 0$$

which is equivalent to the system of double equations

$$\left.\begin{aligned} \beta u + v\alpha + (5\beta - 3\alpha)w &= 0 \\ -\alpha u + \beta v + (3\beta - 5\alpha)w &= 0 \end{aligned}\right\} \qquad (25)$$

Solving (26) by applying the method of cross multiplication, the corresponding non-zero distinct integral solutions to (1) are obtained by

$$\begin{aligned} u(\alpha,\beta) &= 8\alpha^2 - 8\beta^2 - 6\alpha\beta \\ v(\alpha,\beta) &= 2\alpha^2 - 2\beta^2 + 16\alpha\beta \\ w(\alpha,\beta) &= 2\alpha^2 - 2\beta^2 \end{aligned} \qquad (26)$$

Substituting (26) and (15) in (14), the corresponding non-zero integral solution to (1) are given by

$$x(\alpha,\beta) = 8\alpha^2 - 8\beta^2 - 4\alpha\beta$$

$$y(\alpha,\beta) = 2\alpha^2 - 2\beta^2 + 16\alpha\beta$$

$$z(\alpha,\beta) = 2\alpha^2 + 2\beta^2$$

PROPERTIES:

- $x(\alpha,1) + y(\alpha,1) - 10p_\alpha + 10 \equiv 0 \pmod 2$
- $x(\beta+1,\beta) - z(\beta+1,\beta) + 16t_{4,\beta} - 8pr_\beta - 6 = 0$

- $x(\alpha,\alpha)+y(\alpha,\alpha)+z(\alpha,\alpha)-16t_{4,\alpha}=0$
- $x(\alpha,\alpha-1)+z(\alpha,\alpha-1)-16pr_{\alpha}+16t_{4,\alpha}-6=0$
- $x(\alpha-1,\alpha)+y(\alpha-1,\alpha)-5(s_{\alpha}-1)+(GNO_{\alpha})-2t_{4,\alpha}-9=0$

3.5:PATTERN V

One may write (1) as

$$x^2+y^2=17z^2*1 \tag{27}$$

Write 1 as

$$1=\frac{(4+3i)(4-3i)}{25}$$

Assume

$$z=a^2+b^2$$

where a, b are non-zero distinct integers

Using (28) and (3) in (27), we get

$$x^2+y^2=(4+i)(4-i)(a^2+b^2)^2\,\frac{(4+3i)(4-3i)}{5^2}$$

Employing the method of factorization the above equation is written as

$$(x+iy)(x-iy)=(4+i)(4-i)(a+ib)^2(a-ib)^2\,\frac{(4+3i)(4-3i)}{5^2}$$

Equating the positive and negative factors we get,

$$x+iy=\frac{1}{5}(4+i)(4-3i)(a+ib)^2 \tag{29}$$

$$x-iy=\frac{1}{5}(4-i)(4-3i)(a-ib)^2 \tag{30}$$

Equating the real and imaginary part either in (29) or (30), we get

$$x(a,b)=\frac{1}{5}\left(13a^2-13b^2-32ab\right)$$

$$y(a,b)=\frac{1}{5}\left(16a^2-16b^2-26ab\right)$$

As our interest is on finding integer solutions replacing a by 5A and b by 5B, we get

$$\left.\begin{aligned} x(A,B)&=13A^2-13B^2-32AB\\ y(A,B)&=16A^2-16B^2+26AB\\ z(A,B)&=5A^2+5B^2\end{aligned}\right\} \tag{31}$$

Thus (31) and (3) represents non-zero distinct integral solutions of (1)

PROPERTIES:

- $y(A,A)-z(A,A)-16t_{4,A}=0$
- $y(A+1,A)-x(A+1,A)-12t_{3,A}+6t_{4,A}-58pr_{A}-3=0$
- $x(A^2,A)+z(A^{2,}A)-18(t_{4,A})^2+8t_{4,A}+32cp_{A,6}=0$
- $x(B,B-1)+y(B,B-1)+z(B,B-1)-48pr_{B}+38t_{4,B}+29=0$
- $y(A,1)+10t_{4,A}-26pr_{A}+16=0$

REMARK:
Write 1 as

$$1=\frac{(3+4i)(3-4i)}{25} \tag{32}$$

Using (32) and (3) in (1), we get

$$x^2+y^2=(1+4i)(1-4i)(a^2+b^2)\frac{(3+4i)(3-4i)}{5}$$

Employing the method of factorization the above equation is written as

$$x+iy=(1+4i)(a+ib)^2\,\frac{(3+4i)}{5} \tag{33}$$

$$x-iy=(1-4i)(a-ib)^2\,\frac{(3-4i)}{5} \tag{34}$$

Equating the real and imaginary parts either in (33) or (34), we get

$$x(a,b)=\frac{1}{5}\left(-13a^2+13b^2-32ab\right)$$

$$y(a,b)=\frac{1}{5}(-26ab)$$

As our interest is on finding integer solutions replacing a by 5A and b by 5B , we get

$$\left.\begin{aligned} x(A,B) &= -13A^2 + 13B^2 - 32AB \\ y(A,B) &= 1 - 26AB \\ z(A,B) &= 5A^2 + 5B^2 \end{aligned}\right\}$$

GENERATION OF SOLUTIONS:

In this section, we obtain general formula for generating sequences of integer solutions to (1) based on its initial solution.

Formula: 1

Let (x_0, y_0, z_0) be the initial solution to (1)

Let (x_1, y_1, z_1) be the second solution of (1) where

$$x_1 = 3h - x_0 \, , \; y_1 = 3h - y_0 \; , \; z_1 = z_0 + h \qquad (35)$$

be the first solution to (1), where h is the non-zero integer to be determined.

Substituting (35) in (1) and simplifying, we get

$$h = 34z_0 + 6x_0 + 6y_0$$

Substituting (36) in (35), the second solution is obtained as

$$x_1 = 17x_0 + 18y_0 + 102z_0$$

$$y_1 = 18x_0 + 17y_0 + 102z_0$$

$$z_1 = 6x_0 + 6y_0 + 35z_0$$

Expressing the above equations in the matrix form, we have

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = M \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

where $M = \begin{bmatrix} 17 & 18 & 102 \\ 18 & 17 & 102 \\ 6 & 6 & 35 \end{bmatrix}$

Repeating the above process, the general values of x,y and z are given by

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = M \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = M^2 \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

Repeating the above process, the general solution (x_n, y_n, z_n) of (1) based on (x_0, y_0, z_0) is given by

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{bmatrix} = M^{n+1} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

where

$$M^{n+1} = \begin{pmatrix} \frac{y_n + (-1)^n}{2} & \frac{y_n - (-1)^n}{2} & 17x_n \\ \frac{y_n - (-1)^n}{2} & \frac{y_n + (-1)^n}{2} & 17x_n \\ x_n & x_n & y_n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \Rightarrow$$

$$x_{n+1} = \frac{y_n + (-1)^n}{2} x_0 + \frac{y_n - (-1)^n}{2} y_0 + 17x_n z_0$$

$$\Rightarrow \quad y_{n+1} = \frac{y_n - (-1)^n}{2} x_0 + \frac{y_n + (-1)^n}{2} y_0 + 17x_n z_0$$

$$\Rightarrow \quad z_{n+1} = x_n x_0 + x_n y_0 + y_n z_0 \quad , \quad n = 0,1,2\ldots$$

in which (x_n, y_n) represents the general solution of the pellian equation $y^2 = 34x^2 + 1$

4. CONCLUSION

In this paper, we have made an attempt to obtain infinitely many non-zero distinct integer solutions to the equation given by $x^2 + y^2 = 17z^2$. As ternary quadratic equations are rich in variety, one may search for the other choice of ternary quadratic diophantine equations and determine their integer solutions along with suitable properties.

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